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ABSTRACT

The inadequacies in present measurement models are indicated and a description is given of how tree theory, a theory-generative model, overcomes these inadequacies. Among the weaknesses cited in many measurement models are their untested assumptions of linear order and unidimensionality and their inability to generate non-associational relationships for a given set of empirical events. Tree theory is a measurement model whose intent is the generation and testing of specific logical relationships among empirical events. Tree theory has as its basic mathematical framework free Boolean algebra and employs the multinomial distribution as the basis for its statistical tests. The method of tree theory is described in terms of its Boolean algebraic and statistical procedures. Applications are then cited for the fields of evaluation and developmental psychology. (Author/LH)

TREE THEORY: A THEORY-GENERATIVE MEASUREMENT MODEL

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Abstract

The purposes of the paper are to indicate the inadequacies in present measurement models and to describe how tree theory, a theory-generative model, overcomes these inadequacies. Among the weaknesses cited in many measurement models are their untested assumptions of linear order and unidimensionality, and their inability to generate non-associational relationships for a given set of empirical events.

Tree theory is a measurement model whose intent is the generation and testing of specific logical relationships among empirical events. Tree theory has as its basic mathematical framework free Boolean algebra and employs the multinomial distribution as the basis for its statistical tests. The method of tree theory will be described in terms of its Boolean algebraic and statistical procedures. Applications are then cited for the fields of evaluation and developmental psychology.

TREE THEORY: A THEORY-GENERATIVE MEASUREMENT MODEL

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Measurement models in the behavioral sciences are marked by a series of assumptions about the phenomena measured. The first section of this paper identifies a number of these assumptions and indicates their impact on measurement practices. The basic argument made is that the stronger a measurement model is with respect to the assumptions it makes about the traits measured, the less useful for theory generation the model is. The second section of the paper introduces tree theory, a theory-generative measurement model which makes few assumptions about the traits measured. Tree theory is presented in an algebraic and statistical mode. Applications of the theory are cited.

Assumptions of Present Measurement Models

Linear Order Assumption

Most measurement models used in the behavioral sciences assume the existence of a linear order in the trait being measured. For clarity, any property, phenomenon, variable, or entity to be measured will be called a trait. A trait may be further defined as a set of events or manifestations.

Any trait (say T) is linearly ordered if one can define a relation (say R) on the trait with arbitrary constituent manifestations h, k, and

1, with the following properties:

1. assymetric property: hRk implies $k \not{R} h$, where $\not{R}$ means "not R";
2. transitive property: hRk and kRl implies hRl;
3. connected property: either hRk or kRh (Birkhoff & MacLane, 1965).

Examples of linear ordering relations are "is prerequisite to," "is greater than," and "is less than." Any trait on which an assymetric, transitive, and connected relation can be defined is linearly ordered.

Measurement experts such as Brian Ellis (1968) contend that the existence of a quantity implies the existence of a linear order. In classical test theory (Gulliksen, 1950) as well as strong latent trait test theory (Lord and Novick, 1968), linearly ordered traits are assumed. The assumption of a linear ordering of the events comprising a trait is extremely crucial, for it permits measurement to be defined as the assignment of numbers to events. This definition is basic to measurement theory (Campbell, 1928; Stevens, 1946).

A linear order among the events measured makes meaningful the numerical values assigned to the events. The quantification of events is not a neutral procedure. For quantification to be meaningful, it is essential that the relational system among the real numbers mirror the relational system among the events of the trait measured. In the terms of Suppes and Zinnes (1963), an isomorphism must exist between the numerical relational system and the empirical relational system. If such an isomorphism does not exist, then the use of linearly ordered numerals to depict unique events on a non-linearly ordered set of events is inappropriate.

Recent evidence (Bart, 1970) indicates that it is mathematically unreasonable to assume a linear order among a set of events if the set numbers more than four. Instructional planners (Gagne, 1961; Resnick, 1967) and evaluators (Airasian, in press) have shown that the tasks involved in learning many concepts, skills, etc., are frequently nonlinearly ordered. Suppes and Zinnes (1963) have discussed the existence of partial and pseudo orders among classes of psychological phenomena. Thus, a linear order is but one of many possible orders applicable to the description of psychological traits.

Two consequences result from assuming a linear order in a trait when such an order does not empirically exist. First, if a trait is not linearly orderd, then not even an ordinal scale can be attributed to the trait. And if the trait measured does not have at least an ordinal scale, then the use of many common statistical procedures such as correlational analysis, analysis of variance, and the like is suspect.

The second implication involves the broad domain of scores. The most common score index utilized in the behavioral sciences is the total or summative score. Such a score is usually arrived at by summing an individual's correct responses to a set of test items. The use of summative scores can be questioned in its own right. For example, what is the justification for using a linear combination relation as the most appropriate algebraic function relating item scores to trait scores? Also, what is the justification for decomposing performance of a trait into a series of independent subtraits such that each item measures exactly a unit level of the corresponding subtrait? However, summative scores are also closely related to assumptions of trait and item orders, for if a trait or related items are linearly ordered, then one does not need to use summative socres to characterize performance since each item in the order represents a unique level of performance. Two individuals who attain a score of 8 on a linearly ordered set of items attain their scores not necessarily because they answered 8 items correctly, but rather because the highest level item they answered correctly was valued at 8. Further, both individuals have attained identical item response patterns, since for linearly ordered items a given score engenders one and only one response pattern.

Summative scores, which are utilized by most accepted behavioral science measurement models, violate the accepted definition of measurement. Most models define measurement as the assignment of numbers to events (in keeping with the linear order assumption). A summative score violates this definition since a summative score involves the assignment of a number to a class of events.

But more than violating an accepted definition, summative scores cloud the meaning of performance on a trait. Consider two individuals each of whom attains a summative score of 7 on a non-linearly ordered set of test items. Although the scores imply that the individuals are equivalent in performance, such is not necessarily the case, for the individuals may have answered correctly non-overlapping sets of items. Summative scores generally result in the same number being assigned to different sets of events. But because a linear order in the trait measured is assumed (and in spite of the use of summative scores) we are conditioned to thinking that higher scores indicate higher levels of performance. However, in cases where a linear order among the events measured does not exist, a high score is likely to be indicative of a greater breadth of knowledge in the trait rather than indicative of any absolute level of knowledge in the trait.

Consequently, if we are to extricate ourselves from the problems engendered by summative scores and if we are to know whether a score indicates breadth or level of the trait, then we must investigate, not assume, the orders of the traits we measure.

Latent Trait Continuum Assumption

A second assumption made in measurement models is that of an underlying trait continuum (Gulliksen, 1950; Lord & Novick, 1968; Magnussen, 1967; Rasch, 1960). In most measurement we are concerned with the underlying trait which determines performance on an event or a set of events. Most models assume that individuals distribute themselves on a single, underlying continuum with respect to possession of the trait. The finite set of events measuring performance on a trait is assumed to be a sample from the universe of events which could be used to measure the continuum. Further, it is generally assumed that a monotonic relationship exists between degree of trait possession and score on the sample of items. These assumptions result in the use of continuous probability distributions, most frequently the normal, to describe the distribution of individuals with respect to the trait continuum.

We define our traits in terms such as fourth-grade arithmetic achievement, creativity, spatial reasoning, attitude toward school, and so on. To measure, we typically select a sample of items from the universe (usually with some intent to include items near the .5 level of difficulty) and administer our instrument. Individuals who attain high scores (almost always summative in nature) are identified as possessing higher levels of achievement, spacial reasoning, and the like. The trait continuum assumption frequently leads us away from asking such legitimate questions as: (1) Do more items answered correctly truly indicate a higher level of performance or does discussion of level of performance depend upon more precise information about inter-item relationships? (2) Can we define the universe of items clearly enough to justify inferences to a single underlying trait continuum? (3) Is it appropriate to characterize general traits such as creativity and attitude toward school as comprising a single continuum or might there be a number of continua making up these traits?

In sum, the linear order and underlying latent trait continuum assumptions have lead users of accepted measurement models away from consideration of inter-event (or inter-item) relationships. Much information about traits is thereby ignored. It is in consideration of these inter-event relationships, be they linear ordered or not, that the richness for behavioral science model building lies. It is through investigation of such relationships, or the lack of them, that measurement models can be most theory-generative.

Tree Theory

Tree theory is a theory-generative measurement model which makes few assupmtions about the traits measured. In fact, one of the prime aims of tree theory is to provide a framework to enable one to test assumptions of linear order, unidimensionality, the appropriateness of summative scores, and the like.

However, tree theory is capable of more than simply testing assumptions made by other measurement models. With tree theory one can also investigate the accuracy of any a priori postulated order among a set of events used to describe a trait. Further, in those cases where no a priori order among the events is hypothesized, tree theory permits the generation of the specific logical relationships which exist among the events. It is through the investigation of these inter-event relationships that theories about traits can be constructed and tested. The power of tree theory lies in its few assumptions and in its ability to generate and test logical relationships among the events utilized to represent a trait.

To specify the exact logical relationships which exist among a set of events, tree theory utilizes Boolean algebra. Item response patterns are viewed as a form of a free Boolean algebra, with the number of generators in the algebra being equal to the number of items or events studied. The union of the observed item response patterns, as represented in Boolean form, defines the tree for those events or items (i.e., the set of logical relationships among the events or items manifested in the item response patterns).

Specific item parameters, as well as the goodness-of-fit of a specific tree to a set of items, can be determined using a statistical model based upon the multinomial distribution for the item response patterns. The following two sections of the paper describe the Boolean algebraic framework and the rationale for the statistical model used to estimate item parameters and test the goodness-of-fit of specific trees.

Algebraic Procedures

Tree theory is used to determine the logical relationships among empirical events. This task is accomplished through an examination of the item response patterns obtained from a test composed of items measuring the empirical events. Another way to view this process is to state that tree theory reveals the logical relationships among items.

At present, in order to employ tree theory, test items must be dichotomously scored (e.g., a score of "1" is given if an individual "passes" an item and a score of "0" is given if the individuals "fails" an item). Given that one has items which can be dichotomously scored, he can administer the items to a sample of subjects to obtain a set of item response patterns. Every set of distinct item response patterns will determine a unique set of relationships among the items. In fact, there is a one-to-one correspondence between sets of inter-item relationships and sets of distinct response patterns (Bart, 1970).

For a set of items, all of the sets of inter-item logical relationships form a system isomorphic to that of a free Boolean algebra. Any set of logical relationships among a set of items is a tree. Further if one considers the set of distinct response patterns for a set of items, all subsets of that complete set of response patterns also form a system isomorphic to that of a free Boolean algebra. Thus, both the system of sets of logical inter-item relationships and the system of sets of distinct item response patterns have a free Boolean algebraic structure.

In comparing free Boolean algebras with item response pattern sets one can note correspondences between elements in one set and elements in the other set. The dichotomously scored test items have the same properties as generators in a free Boolean algebra. As an item can have two scores, "0" for an incorrect response and "1" for a correct response, so a generator has two states, a negated state and a non-negated state. Thus, for a score of "1" on item i there corresponds the Boolean algebra generator P_i and for a score of "0" on item i there corresponds the generator $-P_i$.

Each item response pattern possible for a set of items has a counterpart element in a free Boolean algebra called an "atom"; any non-null element α in a free Boolean algebra is an atom if, for any element β in the same algebra, either $\alpha \subset \beta$ or $\alpha.\beta = 0$ where "." refers to intersection. Also, as an item response pattern is a function (co-occurrence) of individual item scores, so an atom is a function (intersection) of values of individual generators. For example, the item response pattern (0,1,1) corresponds to the atom $-P_1.P_2.P_3$. Further, as one can generate a set of distinct item response patterns which will determine a tree for the items, so one can generate any element in a free Boolean algebra by forming the union of a certain set of atoms. Thus, because of the isomorphic relationship, many parallels can be cited between response pattern spaces and free Boolean algebras.

To exemplify how the method of tree theory is used, consider the following situation. Let 1, 2, and 3 be three items testing numerical reasoning. Assume that the three items were administered to a sample of subjects and certain distinct response patterns were obtained. Let the obtained response patterns and corresponding Boolean atoms be those indicated in Table 1.

Insert Table 1 about here

To determine the best fitting tree for the items one forms the union of the obtained Boolean atoms, e.g., $-P_1.-P_2.-P_3 v\ P_1.-P_2.-P_3 v\ -P_1.-P_2.P_3 v\ P_1.-P_2.P_3 v\ P_1.P_2.P_3 = P_2 \rightarrow P_1.P_3$ where "$\rightarrow$" refers to "implies", "v" refers to "either ... or", and "." refers to "and." Therefore, the tree for items 1, 2, and 3 indicates that success on item 2 implies success on item 1 and success on item 3. To simplity the union of the obtained atoms as in the example cited to form $P_2 \rightarrow P_1 . P_3$ various algebraic rules proper to Boolean algebra as discussed by Birkhoff and MacLane (1965) are to be employed. In this example, the tree indicated implicative links among the items which could give much information and direction to a teacher or researcher--e.g., the behaviors required for the resolution of item 1 and item 3 must be mastered if behaviors required for the resolution of item 2 are to be mastered. Trees may be represented graphically with branch-like diagrams called tree graphs. For example, $P_2 \rightarrow P_1 . P_3$ is represented graphically in Table 2. In Table 2, P_1 is represented as a prerequisite to P_2, P_3 is represented as a prerequisite to P_2, and P_1 is represented as

Insert Table 2 about here

logically independent of P_3. Thus, success on item 1 is indicated as a prerequisite to success on item 2, success on item 3 is indicated as a prerequisite to success on item 2, and a score on item 1 is indicated as independent of a score on item 3.

For a large number of items the number of trees is substantial. For n items there are 2^n possible item response patterns. Each subset of those 2^n possible item response patterns determines a tree; therefore, there are 2^{2^n} trees possible for a set of n items since there are 2^{2^n} subsets of possible item response patterns. Due to the fact that the number of trees can be gigantic given a large number of items (more than 10), effort are underway to render the method of tree theory into computer language form.

In addition to the empirical tree which is the union of the atoms corresponding to the obtained distinct response patterns for a set of items, there are usually other tenable trees such that only response patterns compatible with these trees are obtained. Other tenable trees for a set of items can be determined by forming the union of the empirical tree and some set of unobtained atoms. Thus, the empirical tree is always included in any tenable tree for a set of items. A response pattern is compatible with a tree if the atom corresponding to the response pattern is included in the tree. Such a pattern is called a confirmatory response pattern. For example, if one considers Table 1, response pattern (0,0,0) is confirmatory with respect to the tree $P_2 \to P_1 . P_3$ for the corresponding atom $-P_1 . -P_2 . -P_3$ is included in the tree $P_2 \to P_1 . P_3$. A response pattern is incompatible with a tree if the atom corresponding to the response pattern is not included in the tree. Such a pattern is called an infirmatory response pattern. For example, in considering Table 1 response pattern (1,1,0) is infirmatory with respect to the tree $P_2 \to P_1 . P_3$ for the corresponding atom $P_1 . P_2 . -P_3$ is not included in the tree $P_2 \to P_1 . P_3$. A tree is confirmed if only confirmatory response patterns are obtained from the appropriate adiminstration of a set of items of a trait under consideration. With the method of tree theory, a researcher could either test specific hypothesized relationships among test items or determine a posteriori specific relationships among test items.

Statistical Procedures

A tree can be generated from either the Boolean algebraic steps just described or from an a priori notion regarding the logical relationships among a set of test items. In either case, one frequently wishes to go beyond the specification of a tree to estimate parameters of particular items in the tree and to test the fit of the estimated parameters to the observed data. This section of the paper presents the simplest, measurement model applicable to estimating the conditional difficulties of items in a tree. Those readers who seek a rigorous derivation of the estimation procedures are referred to Airasian (1970a, 1970b, in press) and Airasian and Bart (1970).

The statistical procedures are based upon two postulates. The first states that, since tree theory is concerned with the discovery of inter-item relationships, response patterns on dichotomously scored items must be utilized as the basic data points. Summative scores are inappropriate indices for studying the logical relationships between items for the reasons cited in the first section of the paper. The second postulate recognizes that the trees studied in a given context are composed of a finite number of test items. The number of resulting response patterns is therefore also finite. As a consequence, a discrete probability distribution is used to describe the distribution of response patterns.

The procedures for estimating conditional item difficulties and concommitant standard errors are based upon two conditions. First, a greater number of confirmatory response patterns than items to be estimated must exist. Second, at least one relationship of implication between the items must be manifested. With these conditions satisfied the general argument for the estimation of the conditional item difficulties and standard errors proceeds as follows.

For n items which are dichotomously scored there exist 2^n different patterns of item responses. However, the tree for a set of items will define one or a number of implication relationships between items which will preclude the occurrence of certain response patterns. Thus if items A and B represent a two-item linearly ordered tree with item B implying item A, the response pattern in which item B is correct but item A is incorrect should not occur.

The Boolean procedures define a tree based upon consideration of each individual's response pattern to the items. Consequently an implication relationship will be defined only when all the data conform to the relationship. One errant response pattern is sufficient to rule out what appears, in all other patterns, to be a hypothesized relationship between two items. In one sense the rigidity of the algebraic procedures is desiralbe because a limitation of the statistical model is that it is deterministic, that is, it can handle only confirmatory response patterns for a given tree. Although the techniques will not be elaborated here, it is possible to handle errant response patterns by Guttman reproducibility procedures (Guttman, 1950; Green, 1956), by specifying a priori tolerance levels for error, or by ignoring those patterns which can be explained on the basis of guessing probabilities.

The distribution of allowable response patterns for a tree can be represented by the multinomial distribution. This distribution represents the generalization of the binomial to cases in which there are greater than two mutually exclusive and exhaustive response categories. The distribution function is

$$\ell = \frac{N!}{n_1!\, n_2! \ldots n_j! \ldots n_J!} P_1^{n_1} P_2^{n_2} \ldots P_j^{n_j} \ldots P_J^{n_J} .$$

where N = total number of subjects with confirmatory response patterns,

n_j = number of subjects manifesting the j-th response pattern,

P_j = probability of the j-th response pattern.

In general, the parameters of interest will be the conditional difficulties for the items in the tree. In the two item linear order example cited above, the parameters would be P(A) and P(B/A). The procedures permit the estimation of more unusual conditional difficulties however, such as the probability of one item being correct given an incorrect response to a prerequisite item. The nature of the item parameters is defined by the nature of the logical relationships between the items in the tree.

Given a tree, the inter-item logical relationships define the response patterns acceptable for that tree. Again referring to the two item linear order, the implicative relationship between A, the lower order item, and B, the higher order item, permit only three of the possible four response patterns. Using a 1 to represent a correct item response and a 0 to represent an incorrect item response (with the first number representing performance on item A), the three allowable response patterns are (00), (10), and (11). Each of these response patterns can in turn be defined in terms of the item parameters to be estimated. Thus, the probability of the pattern (10) is [P(A) (1-P(B/A))].

After expressing the probability of each allowable response pattern for a given tree in terms of the parameters, maximum likelihood estimation procedures can be utilized to derive the parameter estimates. Variance estimates of the conditional difficulty estimates can be found by taking the second derivative of the likelihood function of the response pattern probabilities with respect to each parameter and finding the reciprocal of the negative expectation of the derivative. Standard errors can be calcultated from the variance estimates (Airasian, 1970a, 1970b).

The Boolean algebraic procedures indicate where logical relationships exist among a set of test items. The conditional difficulty estimates provide more precise information about these relationships by indicating the difficulty of an item given performance on all related items. The estimated standard error for each conditional item difficulty gives an indication of the confidence limits of the estimate.

Under certain conditions it is possible to test the fit of the estimated pattern probabilities to the observed pattern probabilities. Such a test indicates the goodness-of-fit of the estimates to the observed data. The condition required to perform such a test is that at least one degree of freedom for the test remain after the estimation of the item parameters. Consider first a linearly ordered tree in which, given two items, one implies the other. For such tree it can be deomnstrated that there exists one more allowable response pattern than items in the order. Thus, if each item parameter is estimated, no degrees of freedom remain to perform the significance test. The fit of a linearly ordered tree is therefore deterministic. If, however, one wishes to test the hypothesis that one or more of the item parameters will have a given value, degrees of freedom which are not used in the estimation of these parameters are available for the significance test.

For other, non-linear trees, sufficient degrees of freedom will remain after the estimation to permit the use of the likelihood ratio statistic. This statistic (λ) is given by the ratio of the maximum of the likelihood in the hypothesized parameter space ω to the maximum of the likelihood in the unrestricted parameter space Ω. Where $L(\omega)$ is the maximum of the likelihood in the restricted space and $L(\Omega)$ is the maximum of the likelihood in the unrestricted parameter space, the likelihood ratio statistic is given by

$$\lambda = \frac{L(\omega)}{L(\Omega)} .$$

For large samples, the critical points of the λ distribution can be approximated by the Chi-Square distribution, where

$$\chi^2 = -2 \log \lambda .$$

The statistical procedures permit estimation of item parameters which take into account the logical relationships between items identified by the Boolean procedures. Confidence limits for these estimates can be derived as a function of the estimated standard errors. Further, a test of the fit of the estimated parameters to the observed data can be performed, given that there exist sufficient degrees of freedom. Such procedures provide more precise information about the relationships between a set of related test items. The estimates can be utilized as a basis for revision of elaboration of given trees (Airasian, in press).

Applications

Applications of tree theory will be cited in two areas, evaluation and developmental psychology. One primary use of tree theory is in placement and formative student evaluation. In placement evaluation (Madaus & Airasian, 1970) one strives to obtain an indication of the extent to which students entering a course have mastered the prerequisites for the course. In formative evaluation, the aim is to identify learning weaknesses and the relationship between these weaknesses prior to grading (Airasian, 1968).

To use the performance on particular prerequisite skills and abilities as a basis for placing a student at a particular point in instruction, one needs to know the relationships between performance on the prerequisites and performance on the course material. It makes little sense to deprive a student of admittance to a course or to attempt to place him somewhere in that course based upon his entry behaviors unless one can define the nature of the specific relationships between entering repertory and the course objectives. For such purposes, summative scores are inadequate. Tree theory, however, can define the set of logical relationships between prerequisites and course performance. The conditional difficulty estimates can provide an index of the likelihood of success on course related tasks given mastery or lack of mastery on prerequisite tasks. Use of tree theory can result in more intelligent and accurate student placement.

Formative evaluation requires that students be afforded specific information about what they have learned and what they have yet to learn at frequent stages of instruction. The aim is to permit correction of learning weaknesses prior to grading or the introduction of subsequent material. Basing a formative test on a tree for a unit can provide students and teachers with an exact indication of what each student has mastered. Further, the types of relationships between the elements to be learned afford an efficient strategy for sequencing correction of unmastered material.

In developmental psychology the determination of orders, patterns, and sequences for cognitive behaviors is an important objective. For example, in Piagetian cognitive stage theory there is the contention that many cognitive processes comply to a linear hierarchy. Tree theory analysis could be used not only to test hypothesized trees (e.g., a linear hierarchy) for various cognitive behaviors but also to determine exacting logical relationships among the cognitive behaviors that will allow revision of developmental psychological theories. In such areas as developmental psychology the discovery potentialities of tree theory should be richly manifested.

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Table 1

Distinct Item Response Patterns and their

Corresponding Atoms obtained for Items 1, 2 and 3.

	Item 1	Item 2	Item 3	Corresponding Atoms
1)	0	0	0	$-P_1 \cdot -P_2 \cdot -P_3$
2)	1	0	0	$P_1 \cdot -P_2 \cdot -P_3$
3)	0	0	1	$-P_1 \cdot -P_2 \cdot P_3$
4)	1	0	1	$P_1 \cdot -P_2 \cdot P_3$
5)	1	1	1	$P_1 \cdot P_2 \cdot P_3$

Table 2

The Tree-Graph for Tree $P_2 \rightarrow P_1 \cdot P_3$